6th Week Wednesday- Vector Spaces

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Recall from physics:

$r \in \mathbb{R}$ → more than this

Linear Algebra

matrices, SLE → Vectorspace ...

Country: $(V, \oplus, \odot)$

Citizens: V a set of vectors

Rules/applications: $\oplus$ → vector addition; $\odot$ → scalar multiplication

Decision (Democratic): 10 properties → VectorSpace ?

Is $(V, \oplus, \odot)$ a vector space or not?

standard vector addition → $(x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+y_2)$ → more than this

standard scalar multiplication → $r.(x_1, y_1) = (rx_1, ry_1)$ → more than this

$\forall r \in \mathbb{R}$, $\vec{v}$

→ n-tupples $n = 1, 2, \ldots, n$

→ matrices $m \times n$

→ polynomials degree less than n

$\mathbb{R}^+$

Ex) $\mathbb{R}^3$, $\oplus$, $\odot$:

→ $\underbrace{(x_1, y_1, z_1)}_{\in\mathbb{R}\ \in\mathbb{R}\ \in\mathbb{R}} \oplus \underbrace{(x_2, y_2, z_2)}_{\in\mathbb{R}\ \in\mathbb{R}\ \in\mathbb{R}} = (\underbrace{x_1 - x_2}_{\in\mathbb{R}}, \underbrace{y_1 + 2y_2}_{\in\mathbb{R}}, \underbrace{z_1 . z_2}_{\in\mathbb{R}}) \in \mathbb{R}^3$

→ $r \odot \underbrace{(x, y, z)}_{\in\mathbb{R}\ \in\mathbb{R}\ \in\mathbb{R}} = (\underbrace{rx}_{\in\mathbb{R}}, \underbrace{ry}_{\in\mathbb{R}}, \underbrace{rz}_{\in\mathbb{R}}) \in \mathbb{R}^3$

$\forall \vec{v} \in V$ $\forall r \in \mathbb{R}$

closed

1) $\underset{\in V}{\vec{v}_1} \oplus \underset{\in V}{\vec{v}_2} \in V$ → closed property of $\oplus$ on V

2) $r \odot \vec{v} \in V$ → " " " $\odot$ on V

Ex) Recall — finite example $V = \{1, 2, 3, 4\}$ $\odot$

Δ	1	2	3	4
1	4	3	1	2
2	2	1	4	3
3	1	2	3	4
4	4	2	1	3

$1 \Delta 2 = 3$

$2 \Delta 1 = 2$

$3 \Delta 1 = 5 \notin V$ not closed

Ex) $\mathbb{R}^2$, $\oplus$, $\odot$

$\underbrace{(x_1, y_1)}_{\in\mathbb{R}^2} \oplus \underbrace{(x_2, y_2)}_{\in\mathbb{R}^2} = (x_1/x_2, \underbrace{y_1 y_2}_{\in\mathbb{R}^2})$ Is $\oplus$ closed on $\mathbb{R}^2$? X

→ $r \odot (x, y) = (rx, ry)$

Counter example: $(3, 5) \oplus (0, -8) = (3/0, -40) \notin \mathbb{R}^2$

undefined $\notin \mathbb{R}$

$\Rightarrow (\mathbb{R}^2, \oplus, \odot)$ is not a vector space.

$\oplus$

1) $\vec{v}_1 \oplus \vec{v}_2 = \vec{v}_2 \oplus \vec{v}_1$ → Commutativity of $\oplus$

2) $\vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3) = (\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3$ → associativity of $\oplus$

identity 3) $\vec{0}_V \in V$ $\vec{v} \oplus \vec{0}_V = \vec{v}$ identity element of $\oplus \in V$

inverse 4) $-\vec{v} \in V$ $\vec{v} \oplus \underset{\in V}{-\vec{v}} = \vec{0}_V$ inverse element of $\oplus$ for any vector $\in V$

Ex) $\mathbb{R}^3$, $\oplus$, $\odot$:

→ $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 - x_2, y_1 + 2y_2, z_1 . z_2)$

→ $r \odot (x, y, z) = (rx, ry, rz)$

$\underbrace{(x_1, y_1, z_1) \oplus (x_2, y_2, z_2)}_{(x_1 - x_2, y_1 + 2y_2, z_1 . z_2)} \overset{?}{=} \underbrace{(x_2, y_2, z_2) \oplus (x_1, y_1, z_1)}_{(x_2 - x_1, y_2 + 2y_1, z_2 . z_1)}$

Counter example:

$(1, 2, 3) \oplus (-1, 3, 4) = (1 - (-1), 2 + 2.3, 3.4) = (2, 8, 12)$

$(-1, 3, 4) \oplus (1, 2, 3) = (-1 - (1), 3 + 2.2, 4.3) = (-2, 7, 12)$ $\neq$

$\oplus$ is not commutative.

$\Rightarrow (\mathbb{R}^3, \oplus, \odot)$ is NOT vector space.

standard: $\mathbb{R}^2$ $(0, 0)$; $\mathbb{R}^3$ $(0, 0, 0)$; matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; poli. 0

$\oplus, \odot$ $\forall r, s \in \mathbb{R}$

1) $r \odot (\vec{v}_1 \oplus \vec{v}_2) = (r \odot \vec{v}_1) \oplus (r \odot \vec{v}_2)$ → Distributivity [D-1]

2) $(r + s) \odot \vec{v} = (r \odot \vec{v}) \oplus (s \odot \vec{v})$ → " [D-2]

3) $(rs) \odot \vec{v} = r \odot (s \odot \vec{v})$ → [D-3]

4) $1 \in \mathbb{R}$ $1 \odot \vec{v} = \vec{v}$ → identity element of $\odot$

Ex $(\mathbb{R}^2, +, \cdot)$

standard $\oplus$ → $(x,y) + (x_2,y_2) = (x_1+x_2, y_1+y_2)$

standard $\odot$ → $r(x,y) = (rx, ry)$

is a vector space.

$\mathbb{R}^3,$ within tripples is a vectorspace

$\mathbb{R}^n, +, \cdot$ within n-tupples is a vector space.

$\mathbb{R}^{m\times n}, \oplus, \odot$

$\oplus$: addition of mxn matrices

$\odot$: scalar multiplication on matrices

is a vector space.

$\mathbb{P}_n$: the set of polynomials of degree less than n

$= \{ a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} : a_i \in \mathbb{R} \}$

$\oplus$: polynomial addition

$\odot$: scalar multiplication on polynomials.

is a vector space.

10. Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by $\mathbb{R}^2$

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$
$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$

$\oplus$ $\odot$

closed
1) $\vec{v}_1 \oplus \vec{v}_2 \in V$ → closed property of $\oplus$ on V ✓ ($\vec{v}_1, \vec{v}_2 \in V$)
2) $r\odot\vec{v} \in V$ → " " " $\odot$ on V ✓

$(x_1, x_2) \oplus (0, 0) = (x_1, x_2)$

$\oplus$

1) $\vec{v}_1 \oplus \vec{v}_2 = \vec{v}_2 \oplus \vec{v}_1$ → Commutativity of $\oplus$ ✓
2) $\vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3) = (\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3$ → associativity of $\oplus$ ✓
identity 3) $\vec{0}_v \in V$ $\quad \vec{v} \oplus \vec{0}_v = \vec{v}$ identity element of $\oplus \in V$ ✗
inverse 4) $-\vec{v} \in V$ $\quad \vec{v} \oplus \underset{\in V}{-\vec{v}} = \vec{0}_v$ inverse element of $\oplus$ for any vector $\in V$ ✗

$\oplus, \odot$ $\forall r,s \in \mathbb{R}$

1) $r\odot(\vec{v}_1 \oplus \vec{v}_2) = (r\odot\vec{v}_1) \oplus (r\odot\vec{v}_2)$ → Distributivity [D-1] ✓
2) $(r+s)\odot\vec{v} = (r\odot\vec{v}) \oplus (s\odot\vec{v})$ → " [D-2] ✗
3) $(rs)\odot\vec{v} = r\odot(s\odot\vec{v})$ → [D-3]
4) $1 \in \mathbb{R}$ $\quad 1\odot\vec{v} = \vec{v}$ → identity element of $\odot$

$r = 2 \quad s = 3 \quad v = (-1, 4)$

$(r+s)\odot\vec{v} = 5\odot(-1,4) = (-5, 20)$

$(r\odot\vec{v}) \oplus (s\odot\vec{v})$

$(-2, 8) \oplus (-3, 12) = (-5, 0)$